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The Log-Expo Inverse Gompertz Distribution: properties and Estimations

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Abstract:

This paper introduces the Log-Expo Inverse Gompertz Distribution (LET-IG) three-parameter distribution and examines some of its mathematical properties. The study derives the density distribution, reliability function, and hazard rate function. It also provides ordinary moments, quantile function, mean residual life, and Renyi entropy. Five estimation methods for the LET-IG distribution based on complete sampling are discussed. A Monte Carlo simulation study is used to calculate the squared bias and variances of the estimates.

Keyword: Inverse Gompertz distribution, Quantile function, mean residual life, Renyi Entropy

1- Introduction

In 2019, researchers Eliwa, El-Morshedy, and Ibrahim [10] introduced the "Inverse Gompertz" distribution, a novel distribution applicable in medical, actuarial, and life sciences research. This distribution is defined with two parameters: a shape parameter and a scale parameter. The cumulative distribution function for a random variable following an Inverse Gompertz distribution is defined as follows:

$$G(x) = e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}, x > 0; \alpha, \beta > 0 \qquad (1)$$

The probability density function is as follows:

$$g(x) = \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}}, x > 0; \alpha, \beta > 0 \qquad (2)$$

M. S. Eliwa (2019) [10] introduced and analyzed a novel three-parameter generalized model called the Kumaraswamy inverse Gompertz distribution. In 2021, M. El-Morshedy [9] developed a four-parameter lifetime model known as the exponentiated generalized inverted Gompertz distribution. In 2022, Arun [2] presented the half Cauchy inverse Gompertz distribution, which utilizes the half-Cauchy distribution as its baseline. In the same year, Moustafa [12] examined the inverse Gompertz distribution (IG) and estimated its survival function. T. M. Adegoke (2023) [13] applied the quadratic rank transmutation map scheme to derive the distribution. Additionally, Taiwo et al. (2023) [14] introduced the Topp-Leone Inverse Gompertz Distribution, an extension of the Gompertz distribution aimed at modeling lifetime datasets. Heba (2023) [4] proposed an adaptive Type-II hybrid progressive censoring strategy to enhance the effectiveness of statistical inference.

2- transformation the log-expo transformation (LET):

Aslam et al. (2020) [4] developed a generator to propose new continuous lifetime distributions defined as:

Let F(x; z) be the cdf of a given random variable depending on some real-valued parameter (s) z. Our approach in this paper consists in enriching this cdf by transforming it into :

$$G(X;\xi) = \frac{log(2 - e^{-\lambda F(x)})}{log(2 - e^{-\lambda})} \quad (3)$$

where ξ = (λ, ζ) for some positive real-valued shape parameter λ and the parameter ζ from the baseline distribution. We call this transformation the log-expo transformation (LET).

The probability density function (pdf) corresponding to Eq (1) is given by

$$g(X) = \frac{\lambda f(x;\nu)e^{-\lambda F(x;\nu)}}{log(2 - e^{-\lambda})(2 - e^{-\lambda F(x;\nu)})} \, x > 0 \quad (4)$$

3-The new distribution LET-IG:

We obtain the PDF and the CDF of LET-inverse Gompertz (LET-IG) by putting (1) and (2) in (3) and (4) as:

$$G(x) = \frac{log(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}})}{log(2 - e^{-\lambda})}, \quad (5)$$

$$g(x) = \frac{\lambda \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}}{log(2 - e^{-\lambda})(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}})} \, x > 0 \quad (6)$$

By the CDF (5) and the PDF (6) we can write the survival function s(x) and the hazard function h(x) as:

$$S(x) = 1 - G(x)$$

$$\therefore S(x) = 1 - \frac{\log\left(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)}{\log(2 - e^{-\lambda})}$$

and,

$$H(x) = \frac{g(x)}{S(x)}$$

$$\therefore H(x) = \frac{\lambda \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)+\frac{\beta}{x}} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}}{\log(2 - e^{-\lambda}) - \log\left(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)}$$

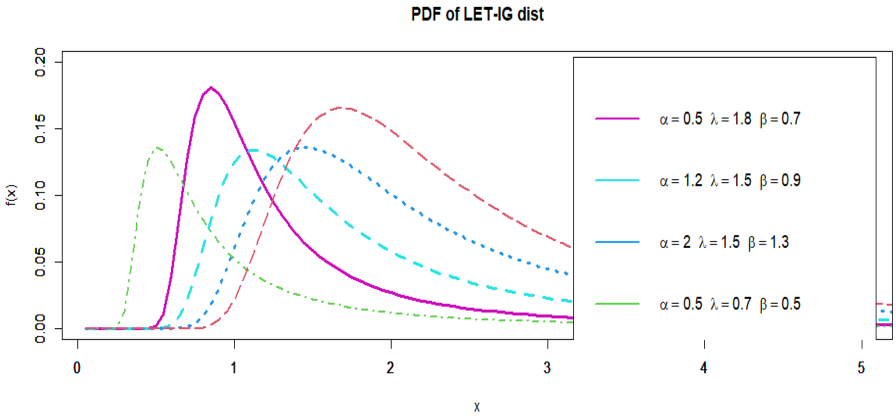

Figure 1. PDF of LET-IG distribution

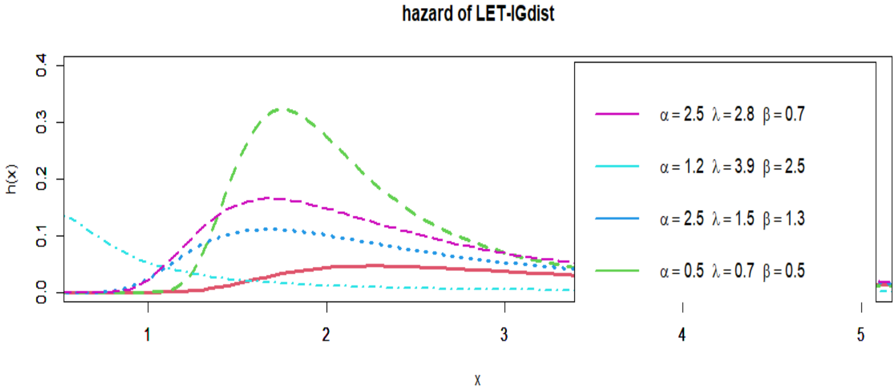

Figure 2.Hazard of LET-IG distribution

4-Quantile function

The Quantile function Q(p) or Q of the LET-IG distribution is the solution of the equation: G(Q)=P. then the quantile function Q at a vector p of percentiles is:

$$Q = \frac{\beta}{\log\left[1 - \frac{\beta}{a}\log t\right]} \tag{7}$$

Where

$$t = \frac{-\log\left(2 - e^{P*\log\left(2-e^{-\lambda}\right)}\right)}{\lambda} \tag{8}$$

We can obtain the generating function of the LET-IG distribution from equations (7,8) by writing x_u instead of Q and u instead of p, where u is the uniform random variable (0,1) then:

$$x_U = \frac{\beta}{\log\left[1 - \frac{\beta}{a}\log t\right]}$$

Where

$$t = \frac{-\log\left(2 - e^{u*\log\left(2-e^{-\lambda}\right)}\right)}{\lambda}$$

The Bowley Skewness measure Bsk and the Moor's Kurtosis measure Mkur [11] are defined by

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

$$Mkur = \frac{Q_{0.875} - 2Q_{0.625} - 2Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

Then the skewness and kurtosis of LET-IG for different parameter

Table.1 Quantile

α, β, λ	$Q_{0.125}$	$Q_{0.25}$	$Q_{0.325}$	$Q_{0.5}$	$Q_{0.625}$	$Q_{0.75}$	$Q_{0.875}$	skewness	kurtosis
0.5;0.7;0.9	0.53625	0.75787	1.30369	0.44931	0.63234	0.94822	2.31902	0.42243	1.54770
0.9; 1.2,1.5	0.87844	1.18798	1.89846	0.75011	1.01519	1.44140	3.15682	0.39307	1.42186
1.2;1.5;0.9	1.22544	1.75058	3.05248	1.02071	1.45270	2.20371	5.48369	0.42515	1.55880
1.5;1.2;2	1.12481	1.53568	2.44429	0.95105	1.30775	1.86433	3.99551	0.37722	1.34483
1.5;2.5;2	1.55892	2.01041	2.96985	1.36132	1.76214	2.36174	4.56189	0.36001	1.27529
1.7;3.7; 5.5	1.76693	2.04938	2.48554	1.61797	1.90275	2.22891	2.98086	0.21392	0.65012
0.9; 0.2;3.2	0.41797	0.58457	0.91593	0.34378	0.49369	0.70948	1.41469	0.33088	1.11514

values are given in the following table.

5-Raw Moments

the r-th raw moment of the LET-IG variable x is

$$\mu_r' = E(x^r)$$

$$\therefore E(x^r) = \int_a^b x^r \frac{\lambda \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)+\frac{\beta}{x}} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}}{\log(2-e^{-\lambda})\left(2-e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)}$$

The mean of x corresponds to r = 1. The mean, variance, skewness and kurtosis of the distribution for various values of the parameters are shown in Table 2. Table 2 indicates that if a, b and λ are fixed, the mean and variance of the LET-IG distribution

Table.2 moment, mean, variance, skewness and kurtosis

α, β, λ	0.7,0.4,4.5	0.9,0.2,3.2	1.5,0.5,2	1.5,0.7,2.9
mean	0.317966	0.264458	0.072827	0.068079
m2	0.462444	0.393213	0.116882	0.102335
m3	0.67257	0.584656	0.187587	0.15383
m4	0.978174	0.869304	0.301064	0.231236
M2	0.361342	0.323275	0.111578	0.097701

CV	1.890507	2.149959	4.586668	4.591329
sk	1.361549	1.684833	4.368642	4.373529
kur	-0.14618	0.838661	17.08505	17.12775

6-Renyi Entropy:

Entropy has been appeared by Alfred Renyi [1] as a logarithmic measure of variation of uncertainty.

If we assume that the events X=$\{x_1, x_2, \ldots\ldots. x_N\}$ have different probability $\{p_1, p_2, \ldots\ldots. p_N\}$. And each deliversbits I_k of information, then the total amount of information for the set is $I_1(p) = \sum_{k=1}^{N} p_k I_k$

Appling the definition to the I(p) we got $\mathrm{I(p)} = g^{-1}\left(\sum_{k=1}^{N} p_k\, g(I_k)\right)$

When the postulate of additively for independent events is applied we get just two possible g(x):

g(x)=cx

$g(x)=c^{-2(1-\alpha)x}$

The first form gives Shannon information and the second gives

$$\mathrm{I}_\alpha(\mathrm{p}) = \frac{1}{1-\alpha} \log \sum_{k=1}^{N} p_k^{\alpha}$$

In continuous distribution $\mathrm{I}_\alpha(\mathrm{p}) = \frac{1}{1-\alpha} \log \int_0^\infty g^{p}(x)\,dx\,, \alpha > 0, \alpha \neq 0$ **i**

Hence,

$$I_\alpha(p) = \frac{1}{1-\alpha} \log \int_0^\infty \left[\frac{\lambda \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)+\frac{\beta}{x}} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}}{\log(2-e^{-\lambda})\left(2-e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)} \right]^{P} dx$$

The following Table below gives the values of Renyi Entropy of LET-IG distribution for different values of the parameters.

Table. 4. Renyi Entropy

α, β, λ	p = 1.5	p =2	p = 4
1.5,0.7,2.9	4.143415	2.9709	2.042445
1.2,1.5,0.9	2.367136	3.5194	2.567032
0.9,0.2,3.2	3.464003	2.316051	1.404314
0.7,0.4,4.5	2.864293	1.763089	0.901075
1.9,1.2,1.5	2.887657	4.261181	-0.73862

It should be noted that the higher the value of the Renyi entropy the greater the level of uncertainty in the system.

7-The mean Residual life:

The mean Residual life (MRL) [15] or the life expectancy at age t is the expected additional life length for a unit, which is alive at age t. the MRL is given by:

$$m(t) = \left(\frac{1}{1-F(x)}\int_t^{\infty} x g(x)dx\right) - t$$

The mean residual life function of LET-IG distribution:

$$\therefore m(t) = \left\{ \frac{1}{1 - \frac{log\left(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)}{log(2-e^{-\lambda})}} \int_t^{\infty} x \frac{\lambda \frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}}{log(2-e^{-\lambda})\left(2 - e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)} dx \right\} - t$$

The following Table below gives the values of MRL of LET-IG distribution for a fixed value of t.

Table. 3. Emperical Mean Residual life . n=50

No	Death.Time	EMRL	No	Death.Time	EMRL
1	0.3282649	1.241909	11	0.5240134	1.327734
2	0.3770292	1.218002	12	0.5319724	1.354506
3	0.3833375	1.237474	13	0.5424948	1.380308
4	0.4410625	1.205396	14	0.5470500	1.413968
5	0.4733350	1.199193	15	0.5482222	1.453161

6	0.4885423	1.210895	16	0.5597515	1.484033
7	0.4924066	1.235101	17	0.5623097	1.526368
8	0.5023810	1.254296	18	0.5668040	1.569432
9	0.5144731	1.272502	19	0.5745460	1.612067
10	0.5237873	1.294767	20	0.5760353	1.664264

8-Methods of Estimation

In this section, we estimate the parameters of the LET-IG distribution using five different methods with a complete sample technique. These methods are: Maximum Likelihood Estimation (MLE), Least-Squares (LS), Weighted Least Squares, Maximum Product of Spacing (MPS), and Percentile-Based Estimation (PE). The performance of each method is evaluated using R software.

8.1-Maximum likelihood estimation (MLE):

The MLE method [8] is a general method and its estimators have some optimum properties such as consistency, asymptotic efficiency and invariance property.

Let $x_1, x_2, \ldots. x_n$ be a random sample from LET-IG population with PDF g(x) given in (8) with unknown parameters α, β, and λ and the log-likelihood function is l(α, β, λ) then the MLE estimates of α, β, and λ are the simultaneously solution of the following equations:

$$\frac{\partial \ell(\alpha,\ \beta,\ \lambda,\theta)}{\partial \alpha}=0,\ \frac{\partial \ell(\alpha,\ \beta,\ \lambda,\theta)}{\partial \beta}=0,\ \frac{\partial \ell(\alpha,\ \beta,\ \lambda,\theta)}{\partial \lambda}=0$$

Which gives the MLE estimates $(\hat{\alpha},\hat{\beta},\hat{\lambda})$. These equations are solved numerically using R software.

$$L(\varphi)=\prod_{i=1}^{n}\frac{\lambda f(x;\varphi)exp\{-\lambda F(x;\varphi)\}}{\log{(2-e^{-\lambda})}(2-exp\{-\lambda F(x;\varphi)\})}$$

And the log-likelihood function by:

$$\log L = nlog\,\lambda + \sum_{i=1}^{n} log\left\{f(x;\varphi) - \lambda \sum_{i=1}^{n} F(x;\varphi)\right\} - nlog\left[\log\left(2 - e^{-\lambda}\right)\right] - \sum_{i=1}^{n} log(2 - exp\{-\lambda F(x;\varphi)\}) \qquad (9)$$

By substituting from equations (3) and (4) as a bassline, in the equation (9) we get:

$$\log L = nlog\,\lambda + \sum_{i=1}^{n} log\left\{\frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}} - \lambda \sum_{i=1}^{n} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right\} - nlog\left[\log\left(2 - e^{-\lambda}\right)\right] - \sum_{i=1}^{n} log\left(2 - exp\left\{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right\}\right) \qquad (10)$$

For obtaining the partial derivatives, differentiating (10) for and we get,

$$\frac{\partial logL}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left[\frac{\sum_{i=1}^{n} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}{\frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}} - \lambda\sum_{i=1}^{n} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right] - \frac{n\left[\frac{e^{-\lambda}}{(2-e^{-\lambda})}\right]}{\log\left(2-e^{-\lambda}\right)} - \sum_{i=1}^{n}\frac{e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} exp\left\{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right\}}{\left(2-exp\left\{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right\}\right)} \qquad (11)$$

$$\frac{\partial logL}{\partial \alpha} = \sum_{i=1}^{n}\frac{\left\{\frac{\alpha}{x^2}\left[\frac{-1}{\beta}\left(\left(e^{\frac{\beta}{x}}-1\right)\right)e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}}\right] + \frac{1}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}}\right\} + \frac{\lambda}{\beta}\sum_{i=1}^{n} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}{\frac{\alpha}{x^2} e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}} - \lambda\sum_{i=1}^{n} e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}} \qquad (12)$$

$$\frac{\partial logL}{\partial\beta}=\sum_{i=1}^{n}\frac{\frac{\left\{\alpha x\left(e^{\frac{\beta}{x}}-1\right)+\beta\left(\beta-\alpha e^{\frac{\beta}{x}}\right)\right\}e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}}}{\beta^{2}x}}{\frac{\alpha}{x^{2}}e^{\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}}-\lambda\sum_{i=1}^{n}e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}$$

$$-\lambda\sum_{i=1}^{n}e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\left[\frac{\alpha\left(e^{\frac{\beta}{x}}-1\right)}{\beta^{2}}-\frac{\alpha e^{\frac{\beta}{x}}}{\beta x}\right]$$

$$-\sum_{i=1}^{n}\frac{\lambda\left[\frac{\alpha\left(e^{\frac{\beta}{x}}-1\right)}{\beta^{2}}-\frac{\alpha e^{\frac{\beta}{x}}}{\beta x}\right]exp\left[-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)\right]}{2-exp\left\{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right\}}\ldots(13)$$

Setting (11), (12) and (13) to zero and solving these equations simultaneously gives the MLE of and i.e., and. However, solving these equations to get the estimates of the unknown parameter is quite difficult. Therefore, a numerical technique such as the newton-raphson method may be used to solve these non-linear equations

8.2-Method of Ordinary Least Squares (L):

The best estimates according to LS method [5] are those which minimize the following quantity:

$$Q_1=\sum_{i=1}^{n}(G(x_{(i)}-\frac{i}{n+1}))^2$$

With respect to α, β and λ.

Where $x_{(i)}$ is the I th orders statistic of LET-IG

8.3-Method of Weighted Least Squares (WLS):

The WLS estimators of α, β, and λ of LET-IG distribution can be obtained by minimizing the quantity:

$$Q_2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{n-i+1}(G(x_{(i)}-\frac{i}{n+1}))^2$$

With respect to α., β, and λ

8.4-Method of Percentile Estimation (PCE):

This method introduced by kao [6, 7] the PCEs estimators of α, β, and λ of LET-IG distribution can be obtained by minimizing the quantity:

$$Q = \sum_{i=1}^{n}[x_{(i)} - a(\frac{1-(1-\sigma)(i/n+1)}{1-(i/n+1)})^{\frac{1}{\sigma^2}}]$$

Where $x_{(i)}$ is the I th orders statistic of LET-IG

8.5-Simulation Study and Data Analysis

The purpose of this section is to compare the performance of the estimation methods discussed in the previous section for the LET-ID distribution, specifically MLE, MPS, LS, WLS, and PE. A Monte Carlo study is conducted to evaluate the behavior of these estimation methods. Additionally, a real data set is analyzed for illustrative purposes. The calculations are performed using the R statistical programming language.

8.5.1-Simulation Study

A simulation study is conducted to compare the performance of the proposed estimation methods using Monte Carlo simulations. The Monte Carlo process involves generating 10000 random samples from the LET-ID distribution under the following conditions:

1. **Sample sizes are**
2. **The following selected parameter cases for α and β of the LET-ID 3rd distribution are assumed:**

a.	$\alpha = 1.5, \beta = 1.7, \lambda = 2.5$
b.	$\alpha = 1.2, \beta = 0.7, \lambda = 1.5$
c.	$\alpha = 2, \beta = 1.9, \lambda = 3$

Based on the generated data and the application of various estimation methods, the mean square errors (MSE) and relative biases (BIAS) for the five different estimation methods are reported in Table 5 through Table 7.

Table.5 The MSE and BIAS for different estimates of the LIT-IG distribution with different values

Estimation	α, β, λ	α = 1.5	β = 1.7	λ = 2.5	α = 1.5	β = 1.7	λ = 2.5
		n =25			n = 50		
MLE	Bias	1.07142	1.81462	1.76854	1.21133	2.07433	2.11894
	MSE	3.12466	2.65934	2.71748	2.88419	4 .65189	6.40684
MPS	Bias	0.12680	0.87820	0.47376	0.06268	0.71736	0.03139
	MSE	3.29141	1.73685	2.07507	1.34885	7.49120	6.90158
LSE	Bias	0.34825	0.06513	0.35330	0.13269	0.07595	0.15264
	MSE	5.51111	1.92656	6.41283	1.71823	1.13543	1.10461
WLSE	Bias	0.25452	0.08338	0.30305	0.07897	0.09443	0.07192
	MSE	3.64872	1.72309	2.20633	1.16322	1.03860	5.15240
PE	Bias	0.38550	0.33948	0.07045	0.26907	0.39566	0.13304
	MSE	5.11247	3.68154	3.35304	3.73763	3.23531	8.29789
		n = 75			n = 100		
MLE	Bias	1.76467	2.42211	2.47572	1.02591	2.94767	2.79403
	MSE	2.67191	3.01254	2.07399	3.78146	5.25004	2.08838
MPS	Bias	0.13027	0.65857	0.10678	0.12586	0.52386	0.10999
	MSE	0.98310	6.13474	4.45835	0.74322	4.57615	3.27246
LSE	Bias	0.03915	0.10232	0.01127	0.02567	0.08958	0.00065
	MSE	1.01136	0.88490	4.07249	0.77579	0.73855	2.81109
WLSE	Bias	0.01036	0.11062	0.01819	0.00079	0.09891	0.02363
	MSE	0.75224	0.80350	2.85239	0.57725	0.67760	2.13815
PE	Bias	0.24164	0.45467	0.20318	0.22884	0.45838	0.23406
	MSE	3.51627	2.76916	6.82446	3.44750	3.06858	6.29763

From the last table:

WLSE has the lowest MSE at larger sample sizes (n = 100), demonstrating the best overall precision. MPS also shows a significant reduction in MSE with increasing sample sizes,

- **particularly at n = 75 and 100, highlighting its effectiveness, for the parameters α.**

  - **WLSE demonstrates the lowest MSE, particularly at larger sample sizes, indicating the best precision overall. LSE also shows a significant reduction in MSE with larger sample sizes, highlighting its effectiveness, for the parameter β.**
- **WLSE and MPS show fluctuating MSE values, indicating inconsistent precision across different sample sizes. LSE shows a significant drop in MSE at n = 50 but fluctuates thereafter, suggesting some variability in precision, for the parameters λ**

Table.6 The MSE and BIAS for different estimates of the LIG distribution with different values

Estimation	α, β, λ	α = 1.2	β = 0.7	λ = 1.5	α = 1.2	β = 0.7	λ = 1.5
		n =25			n = 50		
MLE	Bias	0.39212	0.70428	1.01706	1.97992	0.43251	2.77667
	MSE	2.25312	2.62884	3.90484	2.72072	3.85464	5.62157
MPS	Bias	0.04063	2.34577	0.30268	0.24431	2.07323	0.23322
	MSE	1.73574	8.71595	10.83020	0.92573	6.30030	3.74498
LSE	Bias	0.21485	0.41787	0.29309	0.00525	0.42503	0.00690
	MSE	2.45365	1.58002	9.34259	0.89646	1.02040	3.22668
WLSE	Bias	0.16957	0.42366	0.24948	0.01338	0.45833	0.03203
	MSE	1.79184	1.51593	7.47864	0.71779	1.07303	2.53072
PE	Bias	0.29081	0.06298	0.06305	0.21235	0.01680	0.29459
	MSE	3.36328	4.67765	7.35500	2.89785	5.03805	4.94766
		n = 75			n = 100		
MLE	Bias	1.19976	2.49498	2.68658	0.76504	0.393262	3.056714
	MSE	1.87514	2.45973	5.17503	0.16098	0.0853	7.600119
MPS	Bias	0.23188	1.55874	0.24961	0.25260	0.786547	0.389401
	MSE	0.72917	4.04331	2.87150	0.12988	0.998873	1.527942
LSE	Bias	0.02236	0.25610	0.00472	0.11261	0.034994	0.165438
	MSE	0.67247	0.61420	2.27887	0.11780	0.117295	1.224532
WLSE	Bias	0.00552	0.29796	0.01960	0.06387	0.087206	0.10074
	MSE	0.54514	0.63052	1.94557	0.09658	0.140184	1.089953
PE	Bias	0.11644	0.11101	0.39284	0.01058	0.984877	0.883617

	MSE	**2.48017**	**4.49761**	**4.24812**	**0.59978**	**5.34764**	**2.439193**

From the last table:

- **WLSE and MPS consistently exhibit low MSE values, indicating high precision across all sample sizes. LSE shows a significant decrease in MSE with increasing sample sizes, indicating good precision. MLE and PE show significant decreases in MSE at n = 100, indicating improved precision with larger sample sizes, for the parameters α.**
- **WLSE and LSE consistently exhibit low MSE values, indicating high precision, especially at n = 100, for the parameters β.**
- **WLSE and LSE show consistently low MSE values, particularly at larger sample sizes, indicating high precision. MPS also shows a notable decrease in MSE with increasing sample sizes, indicating improving precision. MLE exhibits increasing MSE, suggesting declining precision with larger sample sizes. PE has relatively high MSE values, indicating moderate precision compared to other methods, for the parameters λ.**

Table.7 The MSE and BIAS for different estimates of the LIG distribution with different values

Estimation	α, β, λ	α = 2	β = 1.9	λ = 3	α = 2	β = 1.9	λ = 3
		n =25			n = 50		
MLE	Bias	1.85497	1.79331	1.43943	1.00398	1.94382	1.82558
	MSE	2.85135	1.18161	4.13203	1.05880	1.11560	3.62018
MPS	Bias	0.11283	0.85083	0.53461	0.05907	0.68972	0.04936
	MSE	5.07697	3.24503	4.77270	1.99101	9.49229	8.23481
LSE	Bias	0.33892	0.09617	0.33926	0.09271	0.11225	0.13114
	MSE	1.23586	2.86830	3.17698	2.60497	1.71181	13.01285
WLSE	Bias	0.22135	0.11886	0.30334	0.04236	0.12324	0.05088
	MSE	6.38453	2.49813	3.71249	1.72775	1.48844	3.48796
PE	Bias	0.27172	0.32992	0.07089	0.12487	0.33664	0.17398

	MSE	7.31721	3.07236	4.89705	4.70727	2.49678	9.35247
		n = 75			n = 100		
MLE	Bias	1.77451	2.07799	2.08411	1.30288	2.01145	2.40088
	MSE	1.18702	2.54362	2.49481	1.75560	2.06201	4.04651
MPS	Bias	0.09880	0.58960	0.04513	0.10035	0.48800	0.06710
	MSE	1.44419	3.40142	5.14416	1.08189	5.84033	3.60118
LSE	Bias	0.01876	0.13421	0.01384	0.00358	0.10991	0.01018
	MSE	1.68728	1.35701	5.62291	1.15178	1.06044	3.46856
WLSE	Bias	0.00410	0.13168	0.01057	0.00778	0.10014	0.02059
	MSE	1.19691	1.19338	3.91868	0.84169	0.88742	2.43030
PE	Bias	0.11709	0.36496	0.22290	0.10353	0.40667	0.25381
	MSE	4.50142	2.60838	7.75243	4.38458	2.61714	7.38117

From the last table:

- **MPS and WLSE show decreasing MSE values, indicating improving precision with larger sample sizes. LSE shows fluctuating MSE values, suggesting variability in precision, for the parameters α.**
- **WLSE consistently shows the lowest MSE, indicating high precision. LSE also shows decreasing MSE with larger sample sizes, indicating improving precision. MPS shows fluctuating MSE, suggesting variability in precision, for the parameters β.**
- **WLSE and LSE demonstrate relatively lower MSE at larger sample sizes, indicating better precision. MPS shows fluctuations in MSE, indicating variability in precision, for the parameters λ.**

8.5.2 -Data Analysis

The tabulated results above reveal several noteworthy observations:

1. **The Mean Square Errors (MSEs) show a decreasing trend as the sample size increases.**

2. **Among the different estimation methods, Maximum Likelihood Estimation (MLE) consistently provides the most accurate estimates for the parameters α and β, while Weighted Least Squares Estimation (WLSE) is the most effective for estimating the parameter λ.**
3. **In terms of the performance ranking of estimators for α and β, MLE is the best, followed by WLSE. For the parameter λ, the ranking is WLSE first, followed by MLE.**

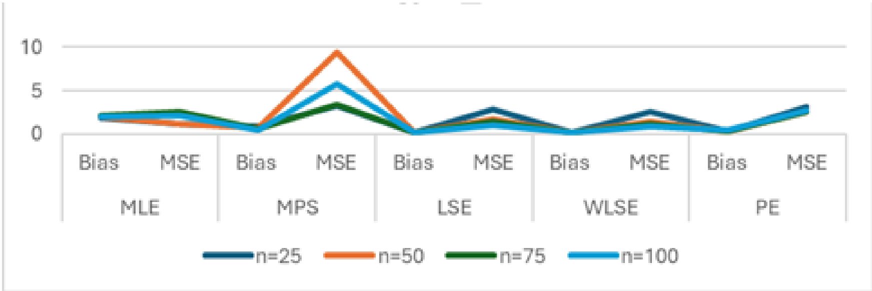

Figure 3.MSE and Bias for parameter α when five estimation methods

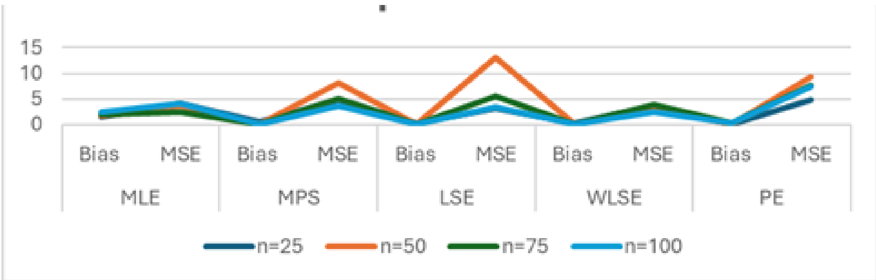

Figure 4.MSE and Bias for parameter β when five estimation methods

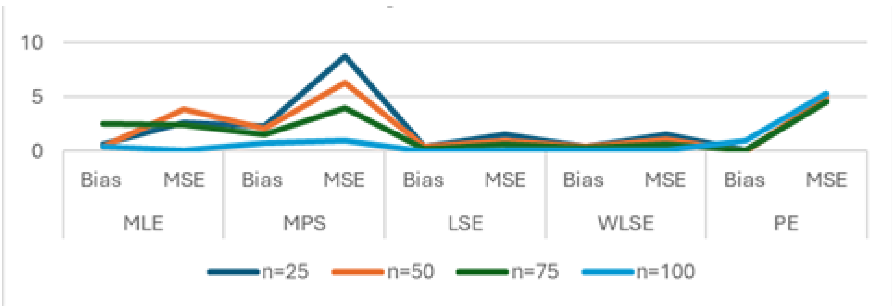

Figure 5.MSE and Bias for parameter λ when five estimation methods

9.Data Analysis and Discussion

In this section, a real data set is utilized to obtain four estimators for the parameters α, β, and λ of the LET-IGD using Maximum Likelihood Estimation (MLE), Maximum Product of Spacings (MPS), Least Squares (LS), and Weighted Least Squares (WLS) methods. The data set, consisting of 134 observations, corresponds to the scores from the General Rating of Affective Symptoms for Preschoolers (GRASP) scale, as collected by M. S. Eliwa et al. [10].

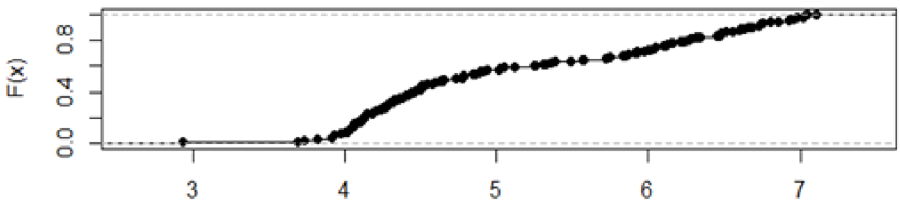

Figure 6.Empirical and theoretical CDF of LET-IG distribution

In order to do this comparison, we compute the MSEs.

Table 11

methods	parameters	α	β	λ
MLE	estimate	1.850660e	4.576917	4.642659
	MSE	1.31487	0.84360	5.66244
MPS	estimate	8.923613	1.751762	41.897848
	MSE	2.34637	1.68555	2.65354
LSE	estimate	0.9635337	15.8515554	2.4069842
	MSE	2.574350	11.918063	4.527349
WLSE	estimate	3.460263	10.735360	7.640079
	MSE	0.3950300	0.5580631	0.6492753

We apply the Kolmogorov– Smirnov (KS) statistic, and its p value in order to verify which estimators of α and β make the LET-IGD fits better to this data.

So, we used the KS test to compare the goodness-of-fit of the LET-IGD with inverse exponential (IG), extended Gompertz (EG) and generalized Gompertz (GG) distributions.

Table 12

Model	α	β	λ	KS	P value
EG	0.003	0.137	1.041	0.241	$3.101 * 10^{-7}$
GG	0.098	0.034	28.902	0.112	0.067
IG	0.154	151.858	---	0.097	0.158
LET-IG	1.850660e	4.576917	4.642659	0.0535	2.129

Also we compute −L, AIC and BIC which are given in Table c. From Tables c, we can conclude that the LET-IGD is the best distribution among all the tested distributions to fit this data

Table 13

Model	−L	AIC	BIC
EG	−442.872	891.744	900.438
GG	−403.171	812.342	821.036
IG	− 393.372	790.744	796.539
LET-IG	-350.07	753.385	758.925

10- Conclusion:

In conclusion, this study introduces the Log-Expo Inverse Gompertz (LET-IG) distribution, detailing its mathematical properties, such as the density distribution, reliability function, hazard rate function, and others. Various estimation methods, including Maximum Likelihood Estimation (MLE), Least Squares (LS), Weighted Least Squares (WLS), Maximum Product of Spacing (MPS), and Percentile-Based Estimation (PE), were explored and their performances compared using Monte Carlo simulations. Based on the findings, it is recommended that further research be conducted to refine these estimation methods and explore their applications in different fields, such as medical and actuarial sciences. Future studies could also consider the development of new estimation techniques to improve the accuracy and efficiency of parameter estimation for the LET-IG distribution.

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