مجلة البحوث المالية والتجارية
المجلد (21) – العدد الثالث – يوليو 2020

Beta Inverted Lindley Distribution: Stress-Strength Reliability &Application

Saeed E. Hemeda
Obour High Institute for Management, Computers & Informatics
Cairo, Egypt
Department of Math & Statistics
saeede@oi.edu.eg

Abstract

This research includes new probability distribution which is named beta inverted Lindley distribution. Some useful functions of the proposed distribution are derived. Important mathematical expansions are investigated. Statistical measures including; quantiles, moments, incomplete moments, Rényi and *s* entropies are acquired. Extra statistical properties such as mean deviations, central of tendency measures, coefficient of variation, coefficients of skewness and kurtosis are defined. The Bonferroni and Lorenz curves are conducted and the stress- strength reliability is computed. The maximum likelihood method is used to estimate parameters of beta inverted Lindley distribution. The importance and significance of the introduced model are applied through failure times data set.

Keywords: **Beta inverted Lindley; Maximum likelihood; Quantile function; Moments; Stress-Strength Reliability.**

1. Introduction

Researchers developed enormous extensions of classical distributions for the modeling of real data using different methods. One of the recent methods is beta-generalized distributions for examples; Eugene *et al.* (2002) introduced the beta-normal, Nadarajah and Kotz (2006) studied beta exponential distribution and Barreto-Souza *et al.* (2010) investigated beta generalized exponential distribution. Newly, Beta-generalized Rayleigh distribution was discussed with applications to lifetime data by Cordeiro *et al.* (2013). Beta-Lindley distribution was determined by Merovci and Sharma (2014) and beta inverted exponential distribution was acquired by Singh and Goel (2015).

The inverted Lindley (IL) distribution is an important probability distribution using in life time data analysis. The IL distribution proposed by Sharma *et al.* (2014). Also, Sharma and Khandelwal (2017) presented new extended inverse Lindley (EIL) and studied its statistical properties.

2. MIL model

The probability and cumulative density functions (pdf and cdf) of beta inverted Lindley (BIL) distribution will be construct. also, survival, hazard and cumulative hazard functions are determined. Further, the asymptotic of pdf, cdf and hazard rate function (hrf) are investigated.
The inverted Lindley (IL) distribution is defined as follows

$$f(x;\theta)=\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}};\; x>0,\, \theta>0, \tag{1}$$

$$G(x;\theta)=\left(1+\frac{\theta}{(1+\theta)x}\right)e^{\frac{-\theta}{x}}. \tag{2}$$

Since, the generalized class of beta distribution is written as

$$F(x;\alpha,\beta,\zeta)=\frac{1}{B(\alpha,\beta)}\int_0^{G(x;\zeta)} x^{\alpha-1}(1-x)^{\beta-1}dx.$$ **(Eugene *et al.* (2002)).**

Then the pdf will be

$$f(x;\alpha,\beta,\zeta)=\frac{1}{B(\alpha,\beta)}\left[G(x;\zeta)\right]^{\alpha-1}\left[1-G(x;\zeta)\right]^{\beta-1}\mathrm{g}(x;\zeta). \tag{3}$$

The cdf of beta-G distribution can be rewritten using the hypergeometric function as follows

$$F(x;\alpha,\beta,\zeta)=\frac{1}{\alpha B(\alpha,\beta)}\left[G(x;\zeta)\right]^{\alpha}\,{}_2\mathrm{F}_1(\alpha,1-\beta,\alpha+1;G(x;\zeta)), \tag{4}$$

If the parameter $\beta > 0$ is a real noninteger, then (5) can be written in series form;

$$F(x;\alpha,\beta,\zeta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(\beta-j)(\alpha+j)\,j!}\left[G(x;\zeta)\right]^{\alpha+j}. \qquad (5)$$

According to beta-G family, the distribution function of BIL distribution is derived by substituting from (1) and (2) into (3) as the following

$$f(x;\Lambda)=\frac{1}{B(\alpha,\beta)}\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha-1}\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^{\beta-1}e^{\frac{-\theta\alpha}{x}}. \qquad (6)$$

Substituting from (2) into (5), the cdf of BIL distribution is obtained as follows

$$F(x;\Lambda)=\frac{e^{\frac{-\theta\alpha}{x}}}{\alpha B(\alpha,\beta)}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)x}\right)e^{-\theta/_x}\right). \qquad (7)$$

Where, Θ is a vector of parameters, $\Lambda=(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$ **and** $\bar{G}(x;\xi)=1-G(x;\xi)$.

$$F(x;\Lambda)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(\beta-i)(\alpha+i)i!}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha+i}e^{\frac{-\theta(\alpha+i)}{x}}. \qquad (8)$$

The reliability $S\,(x;\,\Lambda)$ and hazard $h(x;\,\Lambda)$ functions of BIL distribution are given as follows

$$S(x;\Lambda)=1-\frac{e^{\frac{-\theta\alpha}{x}}}{\alpha B(\alpha,\beta)}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)x}\right)e^{-\theta/_x}\right),$$

$$S(x;\Lambda)=1-\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(\beta-i)(\alpha+i)i!}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha+i}e^{\frac{-\theta(\alpha+i)}{x}}.$$

$$h(x;\Lambda)=\frac{\frac{1}{B(\alpha,\beta)}\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha-1}\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^{\beta-1}e^{\frac{-\theta\alpha}{x}}}{1-\frac{e^{\frac{-\theta\alpha}{x}}}{\alpha B(\alpha,\beta)}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)x}\right)e^{-\theta/_x}\right)} \qquad (9)$$

$$h(x;\Lambda)=\frac{\frac{1}{B(\alpha,\beta)}\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha-1}\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^{\beta-1}e^{\frac{-\theta\alpha}{x}}}{1-\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(\beta-i)(\alpha+i)i!}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha+i}e^{\frac{-\theta(\alpha+i)}{x}}}.$$

Special models from the BIL Distribution

(i) For $\boldsymbol{\alpha}=\boldsymbol{\beta}=\mathbf{1}$, then the BIL distribution reduces to IL distribution.

(ii) For $\boldsymbol{\beta}=\mathbf{1}$, then the BIL reduces to three parameter IL distribution.

(iii) For $\alpha = 1$, the BIL distribution reduces to generalized inverted Lindley (GIL) distribution.

Graphs of pdf and hazard rate function for BIL distribution are represented through Figures 1 and 2.

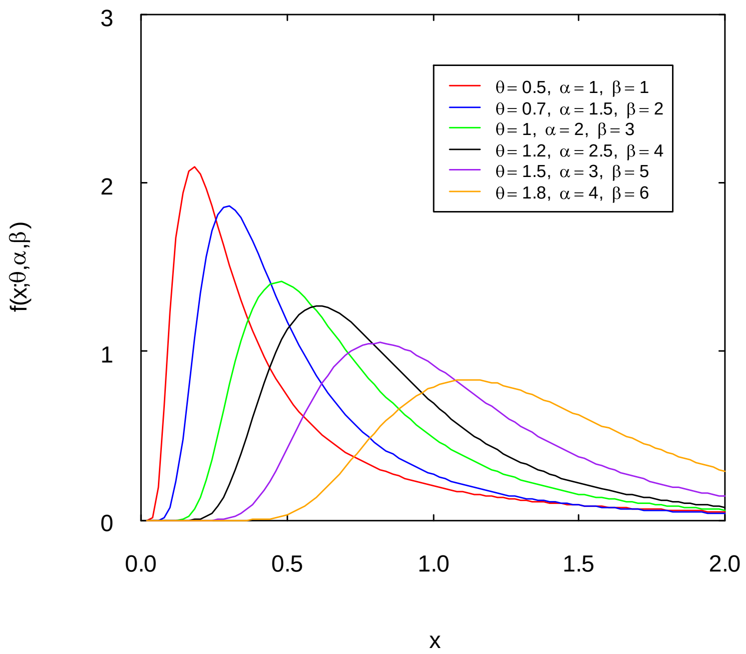

Figure 1: Graph of the pdf of BIL distribution with different parameters values.

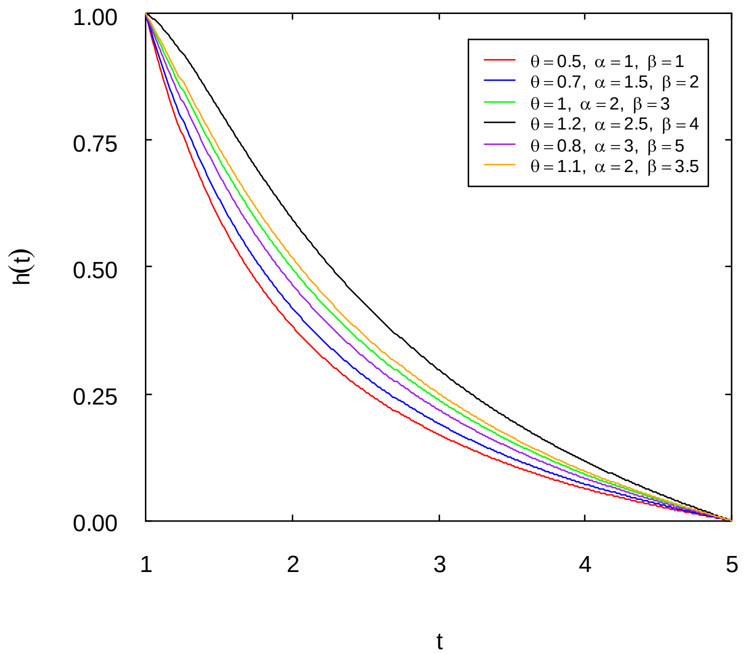

Figure 2: Graph of the hrf of BIL distribution with different parameters values.

3. Important Mathematical Expansions

Expansion forms for the cdf, pdf and reliability are obtained. Using the binomial expansion theorem as follows

$$\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{\frac{-\theta}{x}}\right]^{\beta-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\left(1+\frac{\theta}{(1+\theta)x}\right)^{j}e^{\frac{-\theta j}{x}},$$

By substituting in the pdf (6)

$$f(x;\Lambda)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\frac{1}{B(\alpha,\beta)}\frac{\theta^{2}}{(1+\theta)}\left(\frac{1+x}{x^{3}}\right)\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha+j-1}e^{\frac{-\theta j}{x}}.$$

And

$$\left(1+\frac{\theta}{(1+\theta)x}\right)^{\alpha+j-1}=\sum_{k=0}^{\infty}\binom{\alpha+j-1}{k}\frac{\theta^{k}}{(1+\theta)^{k}x^{k}}.$$

The pdf of BIL can be written in the following expansion

$$f(x;\Lambda)=\frac{1}{B(\alpha,\beta)}\sum_{j,k=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\binom{\alpha+j-1}{k}\frac{\theta^{k+2}}{(1+\theta)^{k+1}}\left[x^{-k-3}e^{\frac{-\theta j}{x}}+x^{-k-2}e^{\frac{-\theta j}{x}}\right].$$

The pdf of BIL takes the form

$$f(x;\Lambda)=\sum_{\mathrm{j},k=0}^{\infty}\Xi_{\mathrm{j,k}}\left[x^{-k-3}e^{\frac{-\theta j}{x}}+x^{-k-2}e^{\frac{-\theta j}{x}}\right]. \qquad (10)$$

Where, $\Xi_{\mathrm{j,k}}=(-1)^{j}\dfrac{1}{B(\alpha,\beta)}\dbinom{\beta-1}{j}\dbinom{\alpha+j-1}{k}\dfrac{\theta^{k+2}}{(1+\theta)^{k+1}}$.

The cdf takes the following series form

$$F(x;\Lambda)=\sum_{i,m=0}^{\infty}\Xi_{i,m}\,x^{-m}e^{\frac{-\theta(\alpha+i)}{x}}. \qquad (11)$$

Where, $\Xi_{i,m}=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\dfrac{(-1)^{i}}{\Gamma(\beta-i)(\alpha+i)i!}\dbinom{\alpha+i}{m}\left(\dfrac{\theta}{1+\theta}\right)^{m}$.

The survival function and hrf , respectively are

$$S(x;\Lambda)=1-\sum_{i,m=0}^{\infty}\Xi_{i,m}\,x^{-m}e^{\frac{-\theta(\alpha+i)}{x}}.$$

$$h(x;\Lambda)=\frac{\sum_{\mathrm{j},k=0}^{\infty}\Xi_{\mathrm{j,k}}\left[x^{-k-3}e^{\frac{-\theta j}{x}}+x^{-k-2}e^{\frac{-\theta j}{x}}\right]}{1-\sum_{i,m=0}^{\infty}\Xi_{i,m}\,x^{-m}e^{\frac{-\theta(\alpha+i)}{x}}}.$$

4. Statistical Properties

Some statistical properties of the BIL distribution, specifically quantile function, moments, moment generating function and mean residual life will be derived.

4.1 Quantile function and Median

The quantile function of the BIL distribution, say, $q(u)=F^{-1}(u)$**, which implies** $F(q(u))=u$**.**

By substituting from equation (7), we obtain

$$\frac{e^{\frac{-\theta\alpha}{q(u)}}}{\alpha B(\alpha,\beta)}\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(u)}\right)e^{-\theta/_{q(u)}}\right)=u,$$

$$-\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}e^{-\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}}=\frac{-p\alpha B(\alpha,\beta)\exp\left\{\frac{-\theta\alpha}{q(u)}-\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}\right\}}{{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(u)}\right)e^{-\theta/_{q(u)}}\right)}.$$

Hence, we have the negative Lambert W_{-1} function of the real argument $\left\{-\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}\right\}$.

$$W_{-1}\left(\frac{p\alpha B(\alpha,\beta)\exp\left\{\frac{-\theta\alpha}{q(p)}-\left(1+\frac{\theta}{(1+\theta)q(p)}\right)^{\alpha}\right\}}{{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(p)}\right)e^{-\theta/_{q(p)}}\right)}\right)=-\left(1+\frac{\theta}{(1+\theta)q(p)}\right)^{\alpha}. \qquad \textbf{(12)}$$

The Lambert W function is defined as the solution of the equation $W(C)\exp(W(C))=z$, where C is a complex number (Jorda (2010)). By solving equation (12) for $q(u)$, we obtain the quantile function.

The median x_{med} of the IBL distribution is derived by substituting $u=0.5$ in equation (12) as follows

$$W_{-1}\left(\frac{0.5\alpha B(\alpha,\beta)\exp\left\{\frac{-\theta\alpha}{q(0.5)}-\left(1+\frac{\theta}{(1+\theta)q(0.5)}\right)^{\alpha}\right\}}{{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(0.5)}\right)e^{-\theta/_{q(0.5)}}\right)}\right)=-\left(1+\frac{\theta}{(1+\theta)q(0.5)}\right)^{\alpha}. \qquad \textbf{(13)}$$

Another way to compute the quantile function is obtained by using equation (7) as follows

$$\frac{e^{\frac{-\theta\alpha}{q(u)}}}{\alpha B(\alpha,\beta)}\left(1+\frac{\theta}{(1+\theta)q(u)}\right)^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(u)}\right)e^{-\theta/_{q(u)}}\right)=u,$$

$$e^{\frac{-\theta\alpha}{q(u)}}\left(1+\frac{\theta}{(1+\theta)(u)}\right)^{\alpha}=\frac{u\,\alpha\,B(\alpha,\beta)}{{}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(u)}\right)e^{-\theta/_{q(u)}}\right)},$$

$$\alpha\ln\left(1+\frac{\theta}{(1+\theta)q(u)}\right)-\frac{\theta\alpha}{q(u)}=\ln\left(u\,\alpha\,B(\alpha,\beta)\right)-\ln\left({}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(u)}\right)e^{-\theta/_{q(u)}}\right)\right)$$

(14)

By solving equation (14) numerically for $q(u)$, we obtain the quantile function of the IBL distribution. Also, the median x_M is derived by substituting $u=0.5$ as follows

$$\alpha\ln\left(1+\frac{\theta}{(1+\theta)q(0.5)}\right)-\frac{\theta\alpha}{q(0.5)}=\ln\left(u\,\alpha\,B(\alpha,\beta)\right)-\ln\left({}_2F_1\left(\alpha,1-\beta,\alpha+1;\left(1+\theta/_{(1+\theta)q(0.5)}\right)e^{-\theta/_{q(0.5)}}\right)\right).$$

(15)

4.2 Moments

Moments are necessary in statistical analysis of various data. It can be used to study the most important characteristics and behaviors of a distribution. In this subsection, the *pth* moment for BILdistribution about zero will be derived. The *pth* moment of random variable *X* can be obtained from (10) as follows

$$\mu_p'=\int_0^{\infty}x^p\ f(x;\Theta)dx,$$

$$\mu_p'=\sum_{j,k=0}^{\infty}\Xi_{j,k}\left[\int_0^{\infty}x^{p-k-3}e^{\frac{-\theta j}{x}}\,dx+\int_0^{\infty}x^{p-k-2}e^{\frac{-\theta j}{x}}\,dx\right],$$

$$\mu_p'=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{j,k}\left[\frac{\Gamma(k-p+2)}{(\theta j)^{k-p+2}}+\frac{\Gamma(k-p+1)}{(\theta j)^{k-p+1}}\right],\qquad \textbf{(16)}$$

$$\Xi^*_{j,k}=(-1)^{j+1}\frac{1}{B(\alpha,\beta)}\binom{\beta-1}{j}\binom{\alpha+j-1}{k}\frac{\theta^{k+2}}{(1+\theta)^{k+1}}.$$

Espetially, the mean *E(X)* and variance *V(X)* of BIL distribution are computed using the 1st and 2nd moments as follows;

$$E(X)=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{i,k}\left[\frac{\Gamma(k+1)}{(\theta j)^{k+1}}+\frac{\Gamma(k)}{(\theta j)^{k}}\right],\qquad \textbf{(17)}$$

$$E(X^2)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{i,k}\left[\frac{\Gamma(k)}{(\theta j)^{k}}+\frac{\Gamma(k-1)}{(\theta j)^{k-1}}\right],$$

$$V(X)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{i,k}\left[\frac{\Gamma(k)}{(\theta j)^{k}}+\frac{\Gamma(k-1)}{(\theta j)^{k-1}}\right]-\left[\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{i,k}\left[\frac{\Gamma(k+1)}{(\theta j)^{k+1}}+\frac{\Gamma(k)}{(\theta j)^{k}}\right]\right]^2.\qquad \textbf{(18)}$$

4.3 Skewness and Kurtosis

Based on (16), the measures of skewness (Sk) and kurtosis (Ku) of BIL distribution can be obtained according to the following relations

$$Sk=\frac{\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3}{\left(\mu_2'-\mu_1'^2\right)^{3/2}},\ Ku=\frac{\mu_4'-4\mu_3'\mu_1'+6\mu_2'\mu_1'^2-3\mu_1'^4}{\left(\mu_2'-\mu_1'^2\right)^2}.$$

Plots of *Sk* and *Ku* of BIL distribution are illustrated through Figure 3 from left to right respectively.

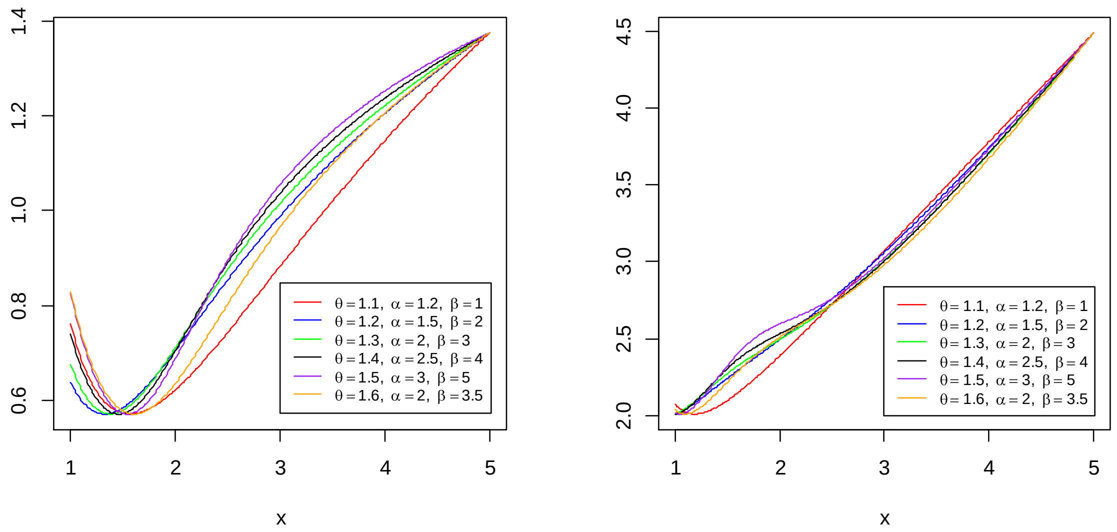

Figure 3: Plots of *Sk* and *Ku* for BIL distribution, respectively.

4.4 Bonferroni and Lorenz curves

Bonferroni (1933) and Lorenz (1905) curves are inequality measures which have useful applications in economics, reliability, demography, actuarial sciences and others. They are defined by

$$Bon(u)=\frac{1}{u\,\mu}\int_0^{q(u)}x\,f(x;\Lambda)dx=\frac{1}{u\,\mu}E(q(u)),\ Lor(u)=\frac{1}{\mu}\int_0^{q(u)}x\,f(x;\Lambda)dx=\frac{1}{\mu}E(q(u)).$$

Where,$0<u\le 1$ and q(u) is the quantile function (Pundir *et al*. (2005)). Therefore, by using equation (14) and (17), the Bonferroni and Lorenz curves are obtained for BIL model.

4.5 Moment Generating & Mean Residual Life Functions

The moment generating function of BIL distribution is obtained by using (16) as follows

$$E_X(t)=\frac{t^p}{p!}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Xi^*_{j,k}\left[\frac{\Gamma(k-p+2)}{(\theta\,j)^{k-p+2}}+\frac{\Gamma(k-p+1)}{(\theta\,j)^{k-p+1}}\right].$$

The mean residual life of BIL ($W(t)$) is defined by

$W(t)=\dfrac{1}{S(t;\Lambda)}\int_t^{\infty} x\, f(x;\Lambda)\,dx-t.$ **(Gupta and Gupta (1983)).**

Using (10), the mean residual life is deduced as

$$W(t)=\frac{\sum_{\mathrm{j},k=0}^{\infty}\Xi_{\mathrm{j},k}\left[\int_t^{\infty}x^{-k-2}e^{\frac{-\theta j}{x}}\,dx+\int_t^{\infty}x^{-k-1}e^{\frac{-\theta j}{x}}\,dx\right]}{1-\sum_{i,m=0}^{\infty}\Xi_{i,m}\,t^{-m}\exp\left\{\frac{-\theta(\alpha+i)}{t}\right\}}-t,$$

$$W(t)=\frac{\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Xi^{*}_{\mathrm{j},k}\left(\frac{\Gamma(k+1,t)}{(\theta j)^{k+1}}+\frac{\Gamma(k,t)}{(\theta j)^{k}}\right)}{1-\sum_{i,m=0}^{\infty}\Xi_{i,m}\,t^{-m}\exp\left\{\frac{-\theta(\alpha+i)}{t}\right\}}-t.$$

Where, $\Gamma(n,t)=\int_t^{\infty}x^{n-1}e^{-x}\,dx$ **is the incomplete gamma function.**

4.6 Rényi and *s*-Entropies

Two entropy measures; Rényi and *s* entropies are studied in this section. Firstly, Rényi entropy of a random variable *X* is defined by

$Ren(\gamma)=\dfrac{1}{1-\gamma}Log\left[\int_0^{\infty}f^{\gamma}(x)\,dx\right],$

where $\gamma>0$ **and** $\gamma\neq1$**. Based on pdf (6), we have**

$$f^{\gamma}(x)=\left(\frac{1}{B(\alpha,\beta)}\right)^{\gamma}\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\left(\frac{1+x}{x^3}\right)^{\gamma}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\gamma(\alpha-1)}\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^{\gamma(\beta-1)}\exp\left\{\frac{-\theta\alpha\gamma}{x}\right\}.$$

Since,

$$\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^{\gamma(\beta-1)}=\sum_{c=0}^{\infty}(-1)^c\binom{\gamma(\beta-1)}{c}\left(1+\frac{\theta}{(1+\theta)x}\right)^{c}e^{\frac{-\theta c}{x}}.$$

$$f^{\gamma}(x)=\sum_{c=0}^{\infty}(-1)^c\binom{\gamma(\beta-1)}{c}\left(\frac{1}{B(\alpha,\beta)}\right)^{\gamma}\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\left(\frac{1+x}{x^3}\right)^{\gamma}\left(1+\frac{\theta}{(1+\theta)x}\right)^{\gamma(\alpha-1)+\mathrm{c}}\exp\left\{\frac{-\theta(\alpha\gamma+c)}{x}\right\}.$$

Also, $\left(1+\dfrac{\theta}{(1+\theta)x}\right)^{\gamma(\alpha-1)+\mathrm{c}}=\sum_{d=0}^{\infty}\binom{\gamma(\alpha\text{-}1)+c}{d}\dfrac{\theta^d}{(1+\theta)^d x^d}.$

Then

$$f^{\gamma}(x)=\sum_{c,d=0}^{\infty}(-1)^c\binom{\gamma(\beta-1)}{c}\binom{\gamma(\alpha\text{-}1)+c}{d}\left(\frac{1}{B(\alpha,\beta)}\right)^{\gamma}\frac{\theta^{d+2\gamma}}{(1+\theta)^{d+\gamma}}\frac{(1+x)^{3\gamma}}{x^{d+3\gamma}}\exp\left\{\frac{-\theta(\alpha\gamma+c)}{x}\right\},$$

Since, $(1+x)^{3\gamma}=\sum_{w=0}^{\infty}\binom{3\gamma}{w}x^{w}$.

Therefore;

$$f^{\gamma}(x)=\sum_{c,d,w=0}^{\infty}\Xi_{c,d,w}\left(\frac{1}{x}\right)^{d+3\gamma-w}\exp\left\{\frac{-\theta(\alpha\gamma+c)}{x}\right\}.$$

Where, $\Xi_{c,d,w}=\sum_{c,d,w=0}^{\infty}(-1)^{c+1}\binom{\gamma(\beta-1)}{c}\binom{\gamma(\alpha-1)+\mathrm{c}}{d}\binom{3\gamma}{w}\left(\frac{1}{B(\alpha,\beta)}\right)^{\gamma}\frac{\theta^{d+2\gamma}}{(1+\theta)^{d+\gamma}}$.

After some calculations, the Rényi entropy takes the form

$$Ren(\gamma)=\frac{1}{1-\gamma}Log\left[\sum_{c,d,w=0}^{\infty}\Xi_{c,d,w}\frac{\Gamma(d+3\gamma-w-1)}{\left[\theta(\alpha\gamma+c)\right]^{d+3\gamma-w-1}}\right],$$

The *s*-entropy, say $Ent(s)$**, is defined by**

$$Ent(s)=\frac{1}{s-1}Log\left[1-\int_{0}^{\infty}f^{s}(x)\,dx\right],$$ **where** $s>0$ **and** $s\neq 1$.

Similarly, one can easily obtain the *s*-entropy of BIL as follows

$$Ent(s)=\frac{1}{s-1}Log\left[1-\sum_{c,d,w=0}^{\infty}\Xi'_{c,d,w}\frac{\Gamma(d+3s-w-1)}{\left[\theta(\alpha s+c)\right]^{(d+3s-w-1)}}\right],$$

Where, $\Xi'_{c,d,w}=\sum_{c,d,w=0}^{\infty}(-1)^{c+1}\binom{s(\beta-1)}{c}\binom{s(\alpha-1)+\mathrm{c}}{d}\binom{3s}{w}\left(\frac{1}{B(\alpha,\beta)}\right)^{s}\frac{\theta^{d+2s}}{(1+\theta)^{d+s}}$.

4.7 Stress-Strength Reliability

Suppose Y and X be independent stress and strength random variables follow beta inverted Lindley distribution with parameters $\Lambda_1=(\theta_1,\alpha_1,\beta_1)$ and $\Lambda_2=(\theta_2,\alpha_2,\beta_2)$, respectively. Then, the stress-strength reliability (SSR) is defined as

$$SSR=P(Y<X)=\int_{0}^{\infty}f(x;\Lambda_1)F(x;\Lambda_2)dx,$$

Substituting from (10) and (11), we deduce that:

$$SSR=\sum_{\mathrm{j},k=0}^{\infty}\sum_{i,m=0}^{\infty}\Xi_{i,m}\,\Xi_{\mathrm{j,k}}\int_{0}^{\infty}\left(x^{-k-m-3}\,e^{\frac{-\theta_1 j-\theta_2(\alpha_2+i)}{x}}+x^{-k-m-2}\,e^{\frac{-\theta_1 j-\theta_2(\alpha_2+i)}{x}}\right)dx.$$

The stress strength reliability of BIL distribution takes the following formula

$$SSR=\sum_{\mathrm{j},k=0}^{\infty}\sum_{i,m=0}^{\infty}\Xi_{i,m}\,\Xi_{\mathrm{j,k}}\left[\frac{\Gamma(k+m+2)}{\left(\theta_1 j+\theta_2(\alpha_2+i)\right)^{k+m+2}}+\frac{\Gamma(k+m+1)}{\left(\theta_1 j+\theta_2(\alpha_2+i)\right)^{k+m+1}}\right].$$

We notice that *SSR* doesn't depend on the value of parameter β.

5. The Maximum Likelihood Estimation

The maximum likelihood estimators of the model parameters $\boldsymbol{\Lambda} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta})$ for beta inverted Lindley distribution and from complete samples are derived as follows:

Let X_1, $X_2, \dots, X_n$ be a simple random sample from beta inverted Lindley distribution with observed values x_1, $x_1, \dots x_n$. Then, the log likelihood function of beta inverted Lindley distribution is obtained as follows

$$ln\,L(x_i;\Lambda) = n\,ln\left[\frac{\theta^2}{(1+\theta)B(\alpha,\beta)}\right] + \sum_{i=1}^{n} ln\left(\frac{1+x_i}{x_i^{\,3}}\right) + (\alpha-1)\sum_{i=1}^{n} ln\left(1+\frac{\theta}{(1+\theta)x_i}\right)$$
$$+(\beta-1)\sum_{i=1}^{n} ln\left[1-\left(1+\frac{\theta}{(1+\theta)x_i}\right)e^{\frac{-\theta}{x_i}}\right] - \frac{\theta\alpha}{\sum_{i=1}^{n} x_i}.$$

Differentiating $lnL(x_i;\Lambda)$ with respect to each parameter and setting the result equals to zero, the maximum likelihood estimators will be obtained.

The maximum likelihood estimates of the model parameters are determined by solving the non-linear equations $\frac{\partial lnL(x_i;\Lambda)}{\partial\alpha} = 0$, $\frac{\partial lnL(x_i;\Lambda)}{\partial\beta} = 0$ and $\frac{\partial lnL(x_i;\Lambda)}{\partial\theta} = 0$. These equations can be solved numerically using some software packages (Mathematica, Mathcad, R, …).

6. Simulation

In this section, a simulation studies are performed to investigate different estimators for BIL model. To check bias and mean square error of the maximum likelihood estimation, random numbers from BIL distribution are generated to perform sampling experiments using Mathematica- software. We obtained the estimators $\hat{\theta}, \hat{\alpha}, \hat{\beta}$ for θ, α, β, respectively. Values of biases and MSEs for these estimates are reported based on 1000 replications with sample size $(n = 25, 50, 100)$ for different selected values of parameters as in Table (1).

The maximum likelihood estimators of the four models are obtained. The performances of the different estimators will be compared through their biases and mean square errors (MSE), for different sample sizes and for different parametric values. The numerical procedures are described through the following steps:

Step (1): A random samples of sizes $n = 25, 50, 100$ are selected and generated from BIL distribution.

Step (2): Different values of the parameters from BIL distribution selected.

Step (3): The parameters have been estimated by the maximum likelihood method for each distribution.

Step (4): 1000 such repetitions are applied to each sample size and for the selected sets of parameters. Biases and mean square errors of different estimators of the parameters are computed using quantile function of BIL distribution as in equation (15). The results are presented in Table (1).

Table 1: Bias and MSE of MLEs for BIL model.

	α =0.1		θ =0.1		β=1		α=1		θ=1.5		β=1	
N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	MSE	Bias	MSE	Bias
25	-0.29	0.366	0.488	20.59	6.085	44.16	20	-0.29	0.366	0.488	20.59	6.085
50	-0.27	0.298	0.257	16.51	4.805	25.44	40	-0.27	0.298	0.257	16.51	4.805
100	-0.05	0.003	-0.1	0.01	-0.83	0.692	100	-0.05	0.003	-0.1	0.01	-0.83
	α =0.1		θ =0.1		β =1		α=1		θ =1		β =1.5	
N	Bias	MSE	Bias	MSE	Bias	MSE	n	Bias	MSE	Bias	MSE	Bias
25	-0.14	0.432	3.463	14.93	-1.49	8.258	20	-0.14	0.432	3.463	14.93	-1.49
50	-0.11	0.695	3.232	12.99	-1.67	5.589	40	-0.11	0.695	3.232	12.99	-1.67
100	-0.05	0.003	-0.1	0.01	-0.83	0.691	100	-0.05	0.003	-0.1	0.01	-0.83
	α =0.1		θ =0.1		β =1		α =1		θ =1		β =1	
N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
25	-0.23	0.4	0.574	8.439	-2.55	13.42	1.648	12.7	-1.63	7.565	0.563	6.145
50	-0.05	0.003	-0.1	0.01	-0.84	0.707	-0.88	0.77	-0.88	0.771	-0.89	0.791
100	-0.05	0.003	-0.1	0.01	-0.84	0.707	-0.88	0.768	-0.88	0.768	-0.89	0.79
	α =0.1		θ =0.1		β =1.1		α =1		θ =1		β =1	
N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
25	-0.3	0.52	0.888	1.035	1.749	17.37	2.414	22.1	-2.43	17.2	1.462	11.31
50	-0.24	0.344	0.36	0.929	1.633	10.77	2.589	18.43	-2.59	17.72	0.65	9.438
100	-0.31	0.57	0.103	0.635	-1.15	7.31	0.201	15.96	-1.57	15.7	-0.15	4.89
	α=0.1		θ =0.1		β =1		α =1.5		θ =1		β =1	
N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
25	-0.23	0.394	0.375	1.088	-0.83	19.47	1.841	2.835	-3.48	33.18	1.712	31.18
50	-0.22	0.393	0.093	0.451	-0.45	17.37	-1.31	1.995	-1.77	9.325	-0.09	14.46
100	-0.05	0.003	-0.1	0.01	-0.1	0.691	0.046	0.715	-0.94	0.888	-0.05	0.72

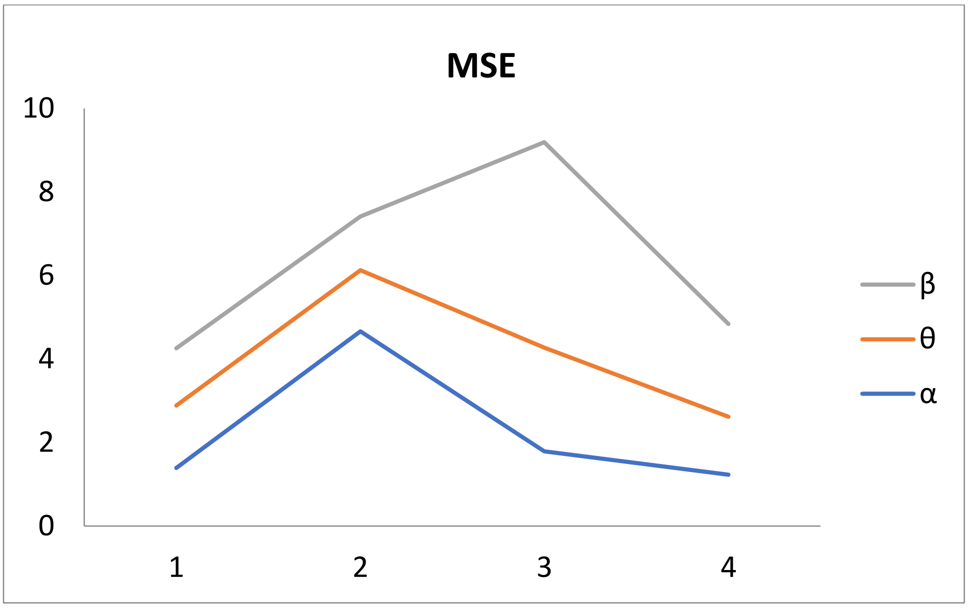

Figure 4: Plots of MSE for BIL distribution with choices parameters values (n=100).

Table 1 and Figure 4 show that: the bias and MSE of the MLE of parameters for BIL distribution decrease as sample size increase.

7. Application to Real Data

In this section, the BIL model fitted to a real data set obtained from Andrews and Herzberg (1985). The data for Kevlar 49/epoxy strands failure times data (pressure at 90 percentage) as follows

0.01	0.01	0.05	0.02	0.02	0.03	0.11	0.04	0.05	0.06	0.07	0.07	0.15
0.01	0.60											
0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24	1.02
0.04	0.85											
0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60	0.63	0.65	0.23
1.03	0.40											
0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	1.26
1.08	1.53											
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33	0.91
0.87	1.02											
1.45	0.90	1.51	0.52	1.53	1.54	0.44	1.55	0.28	1.60	1.43	0.64	0.36	1.85
0.10												

The real data set considered to demonstrate the flexibility of the BIL distribution as compared with some models. For this data set, MLEs of the model parameters are obtained. The model selection is accomplished using the Kolmogorov-Smirnov *(K-S)* statistic and its *p*-value, minus of log-likelihood function $(-2Log\ L)$, Akaike information criterion *(AIC)*, Bayesian information criterion *(BIC)* and the corrected Akaike information criterion *(CAIC)* among the fitted models. The smallest value of each statistic gives the better model as shown in Table 2 In order to assess if the BIL model is appropriate, the histogram of the data, (Akaike (1974) and Schwarz (1978)). the plots of the fitted beta inverse exponential (Singh and Goel (2015)) BIE and IL density functions are displayed in Figure 2. Generally, the smaller value of these statistics is the better fit to real data. The histogram plots and the estimated pdf of the models for each data set are shown. Furthermore, the plots of empirical cdf and estimated cdf of models for the data set are displayed. The numerical results are reported in Table (2). From this table, the following conclusions can be observed.

The pdf of BIE distribution is given by

$$f(x;\alpha,\beta,\theta)=\frac{1}{B(\alpha,\beta)}\frac{\theta}{x^2}e^{\frac{-\alpha\theta}{x}}\left(1-e^{\frac{-\theta}{x}}\right)^{\beta-1};x>0,\ \alpha,\beta,\theta>0.$$

Table (2): Results of fitting different distributions for Kevlar 49/epoxy strands failure times data.

Distribution	MLE of Parameters	*-2logL*	*AIC*	*CAIC*	*BIC*	*K-S*	*P-value*
BIL	$\hat{\alpha}$=1.03 $\hat{\theta}$=5.271 $\hat{\beta}$=1.806	**574.216**	**1018.171**	**1024.435**	**1027.404**	**0.067**	**0.042**
BIE	$\hat{\alpha}$=3.389 $\hat{\theta}$=7.890 $\hat{\beta}$=0.721	**578.712**	**1021.245**	**1026.068**	**1032.400**	**0.074**	**0.036**
IL	$\hat{\theta}$=0.976	**859.802**	**1327.801**	**1422.702**	**1501.813**	**0.168**	**0.014**

From Table 2, we can see that: BIL model is the best fitted model because it has the greatest P-value and smallest K-S statistic value among the other models. BIL distribution is also fitting the data well with smallest values of *-2logL*, *AIC*, *CAIC* and *BIC*.

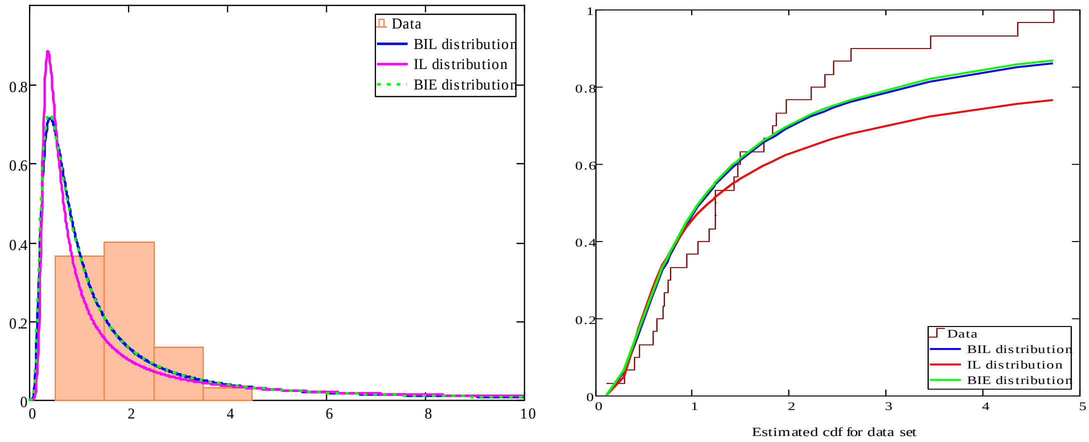

Figure 5: Estimated pdf and cdf of BIL and other compared models

The estimated pdf and cdf plots are shown in Figure 5. From these plots, it can be seen that the estimated density function and cumulative function of BIL model are closely followed the pattern of the histogram and empirical cumulative function of this data set respectively.

8. Conclusion

Beta Lindley inverted probability distribution is presented in this paper. The main aim behind generalization is to make more flexibility to the distribution so that more data can be analyzed using the new distribution. Important statistical measures such as moments, moment generating function, Quantile function and mean deviations are discussed. Furthermore skewness, kurtosis, entropies, Bonferroni and Lorenz curves and stress- strength reliability are determined. The model parameters are estimated using maximum likelihood estimation and simulation study is applied. The importance of the presented distribution is illustrated by using real data set. According to the results; the BIL distribution fits better than BIE and IL distributions to the given data.

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